Approximate Solutions to Complete Elliptic Integrals for Practical Use in Water Engineering

Ali R. Vatankhah

Abstract: Complete elliptic integrals have many applications in water engineering. Examples can be found in different fields such as hydrodynamics, water wave mechanics, groundwater engineering, sediment transport, irrigation and drainage engineering, and shallow water and deep water engineering. In practice, these integrals are evaluated from lookup tables or closed-form solutions (infinite series). Most tables use discrete values of the modulus, which makes accurate interpolation difficult. On the other hand, a series expansion can only be applied to a limited range of the modulus, and is not suitable for manual calculations. Consequently, it is of interest to approximate complete elliptic integrals by simple and accurate algebraic formulas over the entire practical range of the modulus. In current research, the undetermined coefficients method is used for this purpose. Various techniques can be used to determine the unknown coefficients of this method, but most of these do not depend on the integrand. Curve fitting, requiring exact numerical evaluation of the integrals followed by nonlinear optimization, is used as a powerful tool to determine the optimal values of the unknown coefficients as a function of the integrand. The proposed approximations are simple and valid over the full range 0 ≤ k ≤ 0.99 with maximum relative errors of 0.31 and 0.12% for the complete elliptic integrals of the first and second kinds, respectively. More complex approximations achieving higher accuracy with percentage errors lower than 0.07% for 0 ≤ k ≤ 0.99, are also derived. This approach can also be applied to similar integrals of constant integration limits. As such, it should be of interest to practitioners in the water engineering community. DOI: 10.1061/(ASCE)HE.1943-5584.0000376. © 2011 American Society of Civil Engineers.

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Introduction

Many problems in water engineering entail the complete elliptic integrals of the first and second kinds. Some examples are: discharge equation of a circular/elliptical sharp-crested weir (Vatankhah 2010), transient flow nets under hydraulic structures (Ozkan and Adrian 2008), modeling long shore sediment transport under asymmetric waves (Ostrowski and Szmytkiewicz 2006), solving the problem of water withdrawal through a point sink (Hocking et al. 2002), shallow-water equations and undular bores (El et al. 2001), flow fields near continuous wall reactive barriers (Klammler and Hatfield 2008), and design of stable hydraulic sections of an erodible channel (Glover and Florey 1951). At present, the complete elliptic integrals are evaluated using lookup tables, infinite series solutions, approximate solutions, or numerical integration methods. Because of the importance of the complete elliptic integrals in water engineering (in addition to other fields), any effort to improve the computation of these integrals is of practical importance. The objective of this study is to develop simple and accurate approximations to these and other similar integrals using a novel approach based on the method of undetermined coefficients and curve fitting.

Complete Elliptic Integrals K(k) and E(k) and Existing Approximations

Elliptic integrals originally arose from the problem of giving the arc length of an ellipse. Generally, an elliptic integral cannot be expressed in terms of elementary functions. However, every elliptic integral can be transformed into a form that only involves integrals of the first, second, and third kind (Byrd and Friedman 1971).

The complete elliptic integrals of the first and second kinds, K(k) and E(k), respectively, as depicted in Fig. 1, are defined by (Byrd and Friedman 1971)

\[ K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]  
\[ E(k) = \int_{0}^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta \]

where k is called the modulus (k^2 ≤ 1); and θ = integration variable. Both K(k) and E(k) approach \( \pi/2 \) as k approaches 0, while K(k) and E(k) approach infinity and unity, respectively, as k goes to 1. When k = 0 or 1, the integrals can be readily evaluated, otherwise it must be approximated. Because they are important in many areas, a lot of effort has gone into finding efficient ways of calculating elliptic integrals.

Series expansions have been proposed to compute K(k) and E(k) as (Byrd and Friedman 1971; Abramowitz and Stegun 1972)

\[ K(k) = \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1}{2} \times \frac{3}{2} \times 4\right)^2 k^4 + \ldots \right\} \]
\[ + \left(\frac{2n - 1}{}\right)^2 k^{2n} + \ldots \]  

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